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## Weakly pseudocompact subsets of nuclear groups

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### Abstract

Let  $G$  be an Abelian topological group and  $G^+$  the group  $G$  endowed with the weak topology induced by continuous characters. We say that  $G$  respects compactness (pseudocompactness, countable compactness, functional boundedness) if  $G$  and  $G^+$  have the same compact (pseudo-compact, countably compact, functionally bounded) sets. The well-known theorem of Glicksberg that LCA groups respect compactness was extended by Trigos-Arrieta to pseudocompactness and functional boundedness. In this paper we generalize these results to arbitrary nuclear groups, a class of Abelian topological groups which contains LCA groups and nuclear locally convex spaces and is closed with respect to subgroups, separated quotients and arbitrary products. © 1999 Elsevier Science B.V. All rights reserved.

Let  $G$  be an Abelian topological group. By a *character* of  $G$  we mean a homomorphism of  $G$  into the group  $\mathbb{R}/\mathbb{Z}$ . By the *weak topology* on  $G$  we mean the topology induced by the family  $\widehat{G}$  of all continuous characters. It is convenient to denote by  $G^+$  the group  $G$  endowed with its weak topology. We say that  $G$  respects compactness if  $G$  and  $G^+$  have the same compact sets. Similarly, we say that  $G$  respects pseudocompactness (countable compactness, functional boundedness) if  $G$  and  $G^+$  have the same pseudocompact (countably compact, functionally bounded) sets (a subset  $X$  of  $G$  is said to be *functionally bounded* in  $G$  if every real-valued continuous function on  $G$  is bounded on  $X$ ).

That LCA groups respect compactness is a well-known theorem of Glicksberg. It was proved in [6, 7] that LCA groups respect functional boundedness and pseudo-

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compactness. Next, it was proved in [5] that limits of inverse sequences of LCA groups respect compactness. This result was generalized in [3] to arbitrary nuclear groups, a class of Abelian topological groups introduced in [1]. In the present paper we show that nuclear groups respect pseudocompactness, countable compactness and functional boundedness provided that they are complete.

We have to introduce some notation and terminology. Let  $X, Y$  be two symmetric convex subsets of a vector space  $F$  (all vector spaces are assumed to be real). Suppose that  $X \subset Y$ . The *Kolmogorov diameters* of  $X$  with respect to  $Y$  are defined by

$$d_k(X, Y) = \inf_L \inf \{ t > 0 : X \subset tY + L \}, \quad k = 1, 2, \dots,$$

where the infimum is taken over all linear subspaces  $L$  of  $F$  with  $\dim L < k$ .

Let  $G$  be an Abelian topological group. By  $\mathcal{N}_0(G)$  we denote the family of all neighbourhoods of zero in  $G$ . We say that  $G$  is a *nuclear group* if it is separated and satisfies the following condition: given arbitrary  $U \in \mathcal{N}_0(G)$ ,  $c > 0$  and  $m = 1, 2, \dots$ , there exist a vector space  $F$ , two symmetric and convex subsets  $X, Y$  of  $F$  with  $d_k(X, Y) \leq ck^{-m}$  for every  $k$ , a subgroup  $K$  of  $F$  and a homomorphism  $\varphi : K \rightarrow G$  such that  $\varphi(K \cap X) \in \mathcal{N}_0(G)$  and  $\varphi(K \cap Y) \subset U$ . Motivations for introducing nuclear groups and the theory of such groups are presented in [1]. Let us recall here several basic facts; the proofs can be found in Section 7 of [1].

(F<sub>1</sub>) every LCA group is nuclear;

(F<sub>2</sub>) every nuclear locally convex space, treated as an additive topological group, is a nuclear group;

(F<sub>3</sub>) every subgroup of a nuclear group is nuclear;

(F<sub>4</sub>) every separated quotient group of a nuclear group is nuclear;

(F<sub>5</sub>) the product of an arbitrary family of nuclear groups is nuclear;

(F<sub>6</sub>) the direct sum of a countable family of nuclear groups is nuclear;

(F<sub>7</sub>) every group locally isomorphic (in particular, topologically isomorphic) to a nuclear group is nuclear.

Let us also mention that every nuclear group is maximally almost periodic (see Lemma 7 below).

Assertions (F<sub>3</sub>)–(F<sub>6</sub>) say that the permanence properties of nuclear groups are similar to those of nuclear locally convex spaces. In particular, inverse limits of nuclear groups are nuclear.

In connection with (F<sub>2</sub>) it is worth mentioning that if a topological vector space  $E$  is a nuclear group, then  $E$  is a nuclear locally convex space (Proposition (8.9) of [1]). This means, for instance, that if a Banach space  $E$  is a nuclear group, then  $E$  must be finite dimensional. On the other hand, every locally convex space over an ultrametric field is a nuclear group (Proposition (7.11) of [1]).

Let  $F$  be a vector space and  $\tau$  a topology on  $F$  such that  $F_\tau$  is an additive topological group. We say that  $F_\tau$  is a *locally convex vector group* if it is separated and has a base at zero consisting of symmetric convex sets. A locally convex vector group  $F$  is called a *nuclear vector group* if to each symmetric convex  $U \in \mathcal{N}_0(F)$  there corresponds

some symmetric convex  $V \in \mathcal{N}_0(F)$  with  $d_k(V, U) \leq k^{-1}$  for every  $k$ . Every nuclear vector group is a nuclear group (Proposition (9.4) of [1]).

**Lemma 1.** *An Abelian topological group  $G$  is nuclear if and only if there exist a nuclear vector group  $F$ , a subgroup  $H$  of  $F$  and a closed subgroup  $K$  of  $H$  such that  $G$  is topologically isomorphic to  $H/K$ .*

The “if” part follows from  $(F_7)$ ,  $(F_3)$  and  $(F_4)$ . The “only if” part is the main assertion of Theorem (9.6) of [1].

**Lemma 2.** *Let  $\varphi : G \rightarrow H$  be a continuous homomorphism of Abelian topological groups. Then  $\varphi : G^+ \rightarrow H^+$  is continuous, too.*

This simple fact is a direct consequence of the definitions.

**Remark.** Theorem 1.2 of [6] says that if  $G$  and  $H$  are LCA groups, then a homomorphism  $\varphi : G \rightarrow H$  is continuous if and only if  $\varphi : G^+ \rightarrow H^+$  is continuous. The analysis of the proof given in Remark 1.8 of [6] shows that it is enough to assume here that the dual group  $G^\wedge$  respects compactness and that  $G$  is reflexive (i.e. that the canonical mapping  $G \rightarrow G^{\wedge\wedge}$  is a topological isomorphism). That nuclear groups respect compactness was proved in [3]; see also theorem below. A detailed analysis of reflexivity for nuclear groups is given in Ch. 5 of [1]. In particular, countable products of LCA groups and nuclear Fréchet spaces are reflexive, together with their closed subgroups, Hausdorff quotients and dual groups. In general, however, for a nuclear group  $G$  the continuity of  $\varphi : G^+ \rightarrow H^+$  does not imply the continuity of  $\varphi : G \rightarrow H$ . Here is an example.

Let  $H$  be an Abelian topological group with  $H^+ \neq H$  (e.g. let  $H = \mathbb{R}$ ). The group  $G = H^+$  is nuclear, being a subgroup of the Bohr compactification of  $H$  (see  $(F_1)$  and  $(F_3)$ ). Let  $\varphi : G \rightarrow H$  be the identity homomorphism. Then  $\varphi : G^+ \rightarrow H^+$  is a topological identity, while  $\varphi : G \rightarrow H$  is not continuous. It seems to be an interesting question if there exists a similar example with  $G$  nuclear and complete.

Let  $X$  be a subset of a topological space  $S$ . We say that  $X$  is *discretely embedded* in  $S$  if every real-valued function on  $X$  can be extended to a continuous real-valued function on  $S$ . It is clear that every discretely embedded and countable subset of a Hausdorff space must be closed and relatively discrete.

**Lemma 3.** *Let  $X$  be an infinite subset of a discrete Abelian group  $G$ . Then  $X$  contains a denumerable subset which is discretely embedded in  $G^+$ .*

This is a direct consequence of Theorem 1.1.3 of [8].

Let  $A$  be a subset of a normed space  $E$ . The distance of a point  $u \in E$  to  $A$  is denoted by  $d(u, A)$ . The closed unit ball of  $E$  is denoted by  $B_E$ . If  $T : E \rightarrow F$  is a bounded linear operator acting between normed spaces, then we write  $d_k(T) = d_k(T(B_E), B_F)$  for  $k = 1, 2, \dots$ .

**Lemma 4.** Let  $T : E \rightarrow F$  and  $S : F \rightarrow G$  be bounded linear operators acting between pre-Hilbert spaces. Suppose that  $\sum_{k=1}^{\infty} k d_k(T) \leq 1$  and  $d_k(S) \rightarrow 0$  as  $k \rightarrow \infty$ . Let  $K$  be an additive subgroup of  $E$  and  $(a_n)_{n=1}^{\infty}$  a sequence in  $E$  such that

$$d(ST(a_m - a_n), ST(K)) \geq 1, \quad m \neq n.$$

Then one can choose a subsequence  $(a_{n_i})_{i=1}^{\infty}$  of  $(a_n)$  satisfying the following condition: to each  $u \in E$  there corresponds a bounded linear functional  $f$  on  $E$  with  $f(K) \subset \mathbb{Z}$ , such that  $\cos 2\pi f(u - a_{n_i}) \leq \sqrt{2}/2$  for almost all  $i$ .

This is Lemma 3 of [3]. The completeness of the spaces  $E$ ,  $F$  and  $G$ , assumed there, is not essential (cf. the proofs of Lemmas 3 and 4 in [2]).

Let  $p$  be a seminorm on a vector space  $E$ . We write  $B_p = \{u \in E : p(u) \leq 1\}$ . The quotient space  $E/p^{-1}(0)$  endowed with its canonical norm is denoted by  $E_p$ , and the canonical projection of  $E$  onto  $E_p$  by  $\psi_p$ . We say that  $p$  is a *pre-Hilbert* seminorm if  $E_p$  is a pre-Hilbert space. If  $q \leq p$  is another seminorm on  $E$ , then the canonical operator from  $E_p$  to  $E_q$  is denoted by  $T_{pq}$ .

Let  $D$  be a subgroup of an Abelian topological group  $E$ . It is not hard to see that  $D$  is weakly closed in  $E$  if and only if  $E/D$  is maximally almost periodic, i.e. if and only if  $(G/H)^+$  is separated. If  $E$  is a topological vector space, this holds if and only if  $H$  is closed in the weak topology induced on  $E$  by continuous linear functionals (see e.g. Proposition (2.5) of [1]). In the proof of Lemma 5 below by the weak topology we shall always mean the topology induced by continuous characters, even when dealing with subgroups and quotient groups of normed spaces.

**Lemma 5.** Let  $X$  be a subset of a nuclear group  $G$ . If  $X$  is not totally bounded, then it contains a denumerable subset which is discretely embedded in  $G^+$ .

**Proof.** By Lemma 1, we may assume that  $G = H/K$  where  $H$  is a subgroup of some nuclear vector group  $F$ , and  $K$  is a closed subgroup of  $H$ . Consider the canonical diagram

$$\begin{array}{ccc} H & \xrightarrow{\text{id}} & F \\ \downarrow & & \downarrow \gamma \\ H/K & \xrightarrow{\iota} & F/K \end{array}$$

Here  $\iota : H/K \rightarrow F/K$  is a topological embedding, and Lemma 2 implies that  $\iota : (H/K)^+ \rightarrow (F/K)^+$  is continuous. Therefore, without loss of generality, we may assume that  $G = F/K$ .

If  $X$  is not totally bounded, then we can find some  $U \in \mathcal{N}_0(G)$  and some sequence  $(x_n)_{n=1}^{\infty}$  in  $X$  such that  $x_m - x_n \notin U$  if  $m \neq n$ . Choose  $V \in \mathcal{N}_0(F)$  with  $\gamma(V) \subset U$ . As in the proof of Theorem 1 in [2], we can find a linear subspace  $E$  of  $F$  and pre-Hilbert seminorms  $p \geq q \geq r \geq s$  on  $E$  such that  $B_s \subset V$ ,  $B_p \in \mathcal{N}_0(F)$ ,  $\sum_{k=1}^{\infty} k d_k(T_{qr}) \leq 1$ ,

$d_k(T_{pq}) \rightarrow 0$  and  $d_k(T_{rs}) \rightarrow 0$  as  $k \rightarrow \infty$ . We have the canonical diagram

$$\begin{array}{ccccccc} E & \xrightarrow{\text{id}} & E & \xrightarrow{\text{id}} & E & \xrightarrow{\text{id}} & E \\ \downarrow \psi_p & & \downarrow \psi_q & & \downarrow \psi_r & & \downarrow \psi_s \\ E_p & \xrightarrow{T_{pq}} & E_q & \xrightarrow{T_{qr}} & E_r & \xrightarrow{T_{rs}} & E_s \end{array}$$

Since  $d_k(T_{pq}) \rightarrow 0$  as  $k \rightarrow \infty$ , and  $T_{pq}$  maps  $E_p$  onto  $E_q$ , it follows that  $E_q$  is separable. Let  $H = E \cap K$  and let  $D$  be the weak closure of  $\psi_q(H)$  in  $E_q$ . Consider the canonical diagram

$$\begin{array}{ccccc} E_q & \xleftarrow{\psi_q} & E & \xrightarrow{\text{id}} & F \\ \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma \\ E_q/D & \xleftarrow{\mu} & E/H & \xrightarrow{\nu} & F/K \end{array}$$

The subspace  $E$  spanned over  $B_p$  is an open subgroup of  $F$  because  $B_p \in \mathcal{N}_0(F)$ . Hence  $A := \gamma(E)$  is an open subgroup of  $G = F/K$ . Observe that, since  $E$  is an open subgroup of  $F$ ,  $\nu$  is a topological embedding. By Lemma 2, the canonical projection  $\pi : G^+ \rightarrow (G/A)^+$  is continuous.

Suppose first that the set  $S = \{\pi(x_n)\}_{n=1}^\infty$  is infinite. Then, by Lemma 3, it contains a denumerable subset  $P$  which is discretely embedded in  $(G/A)^+$ . For each  $p \in P$ , choose some  $n_p$  such that  $\pi(x_{n_p}) = p$ . Then it is clear that the set  $\{x_{n_p}\}_{p \in P}$  is denumerable and discretely embedded in  $G^+$ .

Next, suppose that  $S$  is finite. Then we can choose a subsequence  $(x'_n)_{n=1}^\infty$  of  $(x_n)$  such that  $\pi(x'_n)$  is constant. Consequently, we can choose a sequence  $(a_n)_{n=1}^\infty$  in  $E$  such that  $x'_n - x'_1 = \gamma(a_n)$  for all  $n$ . Since  $x'_m - x'_n \notin U$  whenever  $m \neq n$ , we have that

$$d(T_{rs}T_{qr}(\psi_q(a_m) - \psi_q(a_n)), T_{rs}T_{qr}(\psi_q(H))) \geq 1.$$

Then it follows directly from Lemma 4 that we can choose a subsequence  $(a_n)_{i=1}^\infty$  of  $(a_n)$  such that the sequence  $(\alpha\psi_q(a_{n_i}))_{i=1}^\infty$  does not have weak cluster points in  $E_q/D$ . Then the set  $Z = \{\alpha\psi_q(a_{n_i})\}_{i=1}^\infty$  is relatively discrete and closed in  $(E_q/D)^+$ . Without loss of generality, we may assume that  $\alpha\psi_q(a_{n_i}) \neq \alpha\psi_q(a_{n_j})$  if  $i \neq j$ . To complete the proof, it is enough to show that the set  $Y = \{\gamma(a_{n_i})\}_{i=1}^\infty$  is discretely embedded in  $(F/K)^+$ .

Let  $\xi$  be an arbitrary real-valued function on  $Y$ . Consider the function  $\eta : Z \rightarrow \mathbb{R}$  given by  $\eta(\alpha\psi_q(a_{n_i})) = \xi(\gamma(a_{n_i}))$  for every  $i$ . Since  $E_q$  was a separable normed space, the group  $E_q/D$  is separable and metrizable, hence Lindelöf. The group  $(E_q/D)^+$  is a continuous image of  $E_q/D$ , therefore it is Lindelöf, too.  $(E_q/D)^+$  is separated, hence completely regular, because  $D$  was weakly closed in  $E_q$ . Thus  $(E_q/D)^+$  is a normal space. So, we can extend  $\eta$  to a continuous function  $\zeta : (E_q/D)^+ \rightarrow \mathbb{R}$ .

Being a vector space,  $E$  is a divisible group, so that the group  $A = \gamma(E) = \nu(E/H)$  is divisible, too. Consequently, the identity homomorphism  $A \rightarrow A$  can be extended to some homomorphism  $\sigma : G \rightarrow A$  (see [4], (A.7)). By Lemma 2, the homomorphisms  $\sigma : G^+ \rightarrow A^+$ ,  $\nu^{-1} : A^+ \rightarrow (E/H)^+$  and  $\mu : (E/H)^+ \rightarrow (E_q/D)^+$  are continuous. Then the function  $\zeta\mu\nu^{-1}\sigma : G^+ \rightarrow \mathbb{R}$  is a continuous extension of  $\xi$ .  $\square$

We shall identify  $\mathbb{R}/\mathbb{Z}$  with the interval  $I = (-\frac{1}{2}, \frac{1}{2}]$  and treat characters as functions with values in  $I$ . Let  $\chi$  be a character of an Abelian group  $G$  and let  $A$  be a subset of  $G$ . We write

$$|\chi(A)| = \sup\{|\chi(g)| : g \in A\}.$$

By  $2\chi$  we denote the character of  $G$  given by  $(2\chi)(g) = \chi(g + g)$  for  $g \in G$ . The following simple fact is a direct consequence of definitions.

**Lemma 6.** *If  $|\chi(A)| \leq \frac{1}{4}$  and  $|(2\chi)(A)| \leq \frac{1}{4}$ , then  $|\chi(A)| \leq \frac{1}{8}$ .*

Let  $G$  be an abelian topological group. The group of all continuous characters of  $G$  is denoted by  $G^\wedge$ . A subset  $A$  of  $G$  is said to be *quasi-convex* if to each  $g \in G \setminus A$  there corresponds some  $\chi \in G^\wedge$  with  $|\chi(A)| \leq \frac{1}{4}$  and  $|\chi(g)| > \frac{1}{4}$ . We say that  $G$  is a *locally quasi-convex* group if it has a base of neighbourhoods of zero consisting of quasi-convex sets. It is clear that every separated locally quasi-convex group is maximally almost periodic.

**Lemma 7.** *Every nuclear group is locally quasi-convex.*

This is Theorem (8.5) of [1].

**Lemma 8.** *The completion of a locally quasi-convex group is locally quasi-convex.*

**Proof.** Let  $\tilde{G}$  be the completion of a locally quasi-convex group  $G$ . We may identify  $G$  with a dense subgroup of  $\tilde{G}$ . Given a subset  $A$  of  $G$ , by  $\bar{A}$  we shall denote the closure of  $A$  in  $\tilde{G}$ . For each  $\chi \in G^\wedge$ , let  $\tilde{\chi} \in (\tilde{G})^\wedge$  be the canonical extension of  $\chi$ .

Choose an arbitrary  $U \in \mathcal{N}_0(\tilde{G})$ . We have to find a quasi-convex neighbourhood of zero in  $\tilde{G}$  contained in  $U$ . Since  $U \in \mathcal{N}_0(\tilde{G})$ , there is some  $V \in \mathcal{N}_0(G)$  with  $\bar{V} \subset U$ . We may assume that  $V$  is a quasi-convex subset of  $G$ . Next, we can find some  $W \in \mathcal{N}_0(G)$  with  $W + W \subset V$ . Let us denote

$$V^0 = \{\chi \in G^\wedge : |\chi(g)| \leq \frac{1}{4} \text{ for each } g \in V\},$$

$$W^0 = \{\chi \in G^\wedge : |\chi(g)| \leq \frac{1}{4} \text{ for each } g \in W\},$$

$$S = \{g \in \tilde{G} : |\tilde{\chi}(g)| \leq \frac{1}{4} \text{ for each } \chi \in W^0\}.$$

It is clear that  $S$  is a closed and quasi-convex subset of  $\tilde{G}$  containing  $W$ . Since  $W \in \mathcal{N}_0(G)$ , we have  $\bar{W} \in \mathcal{N}_0(\tilde{G})$  and therefore  $S \in \mathcal{N}_0(\tilde{G})$ . So, to complete the proof it is enough to show that  $S \subset \bar{V}$ .

Take an arbitrary  $\chi \in V^0$ . Then we have

$$|\chi(W)|, |(2\chi)(W)| \leq |\chi(W + W)| \leq |\chi(V)| \leq \frac{1}{4}.$$

Thus  $\chi \in W^0$  and  $2\chi \in W^0$ . Hence  $|\widetilde{\chi}(S)| \leq \frac{1}{4}$  and  $|(2\widetilde{\chi})(S)| = |(\widetilde{2\chi})(S)| \leq \frac{1}{4}$ . By Lemma 6, this implies that  $|\widetilde{\chi}(S)| \leq \frac{1}{8}$ . Hence,

$$|\chi((S+S) \cap G)| = |\widetilde{\chi}((S+S) \cap G)| \leq |\widetilde{\chi}(S+S)| \leq |\widetilde{\chi}(S)| + |\widetilde{\chi}(S)| \leq \frac{1}{4}.$$

Since this holds for any  $\chi \in V^0$ , and  $V$  is quasi-convex, it follows that

$$(S+S) \cap G \subset V. \quad (1)$$

For each  $P \in \mathcal{N}_0(\widetilde{G})$  one has  $S \subset \overline{(S+P) \cap G}$  because  $G$  is dense in  $\widetilde{G}$ . Setting here, in particular,  $P = S$  and applying (1), one gets  $S \subset \overline{V}$ .  $\square$

**Lemma 9.** *Let  $X$  be a totally bounded subset of a nuclear group. Then the weak topology on  $X$  is equal to the original one.*

**Proof.** Denote the nuclear group by  $G$ . We may identify it with a dense subgroup of its completion  $\widetilde{G}$ . Being nuclear,  $G$  is separated, and therefore so is  $\widetilde{G}$ . Next,  $G$  is locally quasi-convex due to Lemma 7 and therefore, by Lemma 8,  $\widetilde{G}$  is locally quasi-convex, too. Thus  $\widetilde{G}$  is maximally almost periodic, which means that the canonical homomorphism  $\varphi : \widetilde{G} \rightarrow \mathbb{T}^{(\widetilde{G})^\wedge}$  given by  $\varphi(g)(\chi) = \chi(g)$  for  $\chi \in (\widetilde{G})^\wedge$  is injective. Observe that  $\varphi$  is a homeomorphism of  $\widetilde{G}^+$  onto its image in  $\mathbb{T}^{(\widetilde{G})^\wedge}$ . Observe also that the topology of  $G^+$  is equal to the topology induced on  $G$  by the embedding of  $G$  into  $\widetilde{G}^+$ .

Since  $X$  is totally bounded, its closure  $\overline{X}$  in  $\widetilde{G}$  is compact. Consequently,  $\varphi|_{\overline{X}} : \overline{X} \rightarrow \varphi(\overline{X})$  and  $\varphi|_X : X \rightarrow \varphi(X)$  are homeomorphisms.  $\square$

**Theorem.** *Nuclear groups respect compactness, countable compactness and pseudo-compactness. Complete nuclear groups respect functional boundedness.*

The authors do not know if the assumption of completeness can be removed.

**Proof.** Let  $X$  be a subset of a nuclear group  $G$ . If  $X$  is compact, countably compact, pseudocompact or functionally bounded in the weak topology, then  $X$  is totally bounded in the original topology, due to Lemma 5. Consequently, by Lemma 9, the weak topology on  $X$  is equal to the original one. If, in addition,  $G$  is complete, then  $\overline{X}$  is compact and hence  $X$  is functionally bounded.  $\square$

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